Symmetric truncations of the shallow-water equations

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Conservation of potential vorticity in Eulerian fluids reflects *particle interchange symmetry* in the Lagrangian fluid version of the same theory. The algebra associated with this symmetry in the shallow-water equations is studied here, and we give a method for truncating the degrees of freedom of the theory which preserves a maximal number of invariants associated with this algebra. The finite-dimensional symmetry associated with keeping only N modes of the shallow-water flow is SU(N). In the limit where the number of modes goes to infinity ($N\rightarrow\infty$) all the conservation laws connected with potential vorticity conservation are recovered. We also present a Hamiltonian which is invariant under this truncated symmetry and which reduces to the familiar shallow-water Hamiltonian when $N\rightarrow\infty$. All this provides a finite-dimensional framework for numerical work with the shallow-water equations which preserves not only energy and enstrophy but all other known conserved quantities consistent with the finite number of degrees of freedom. The extension of these ideas to other nearly two-dimensional flows is discussed.

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I. INTRODUCTION

Many geophysical problems are naturally decomposed into a many-layered approximation with each layer governed by the shallow-water equations [1]. These equations take the fluid density to be consistent in each layer, and because the horizontal dimensions are assumed to be much larger than the vertical ones, hydrostatic balance is taken to hold in each layer separately. Vertical variations in each layer are ignored in the two-dimensional horizontal velocity $\mathbf{v}(\mathbf{x},t)=[v_1(x,y,t),v_2(x,y,t)][\mathbf{x}=(x,y)]$ and incompressibility

$$\nabla\cdot\mathbf{v}(x,t)+\frac{\partial v_3(\mathbf{x},z,t)}{\partial z}=0 \tag{1}$$

determines the vertical velocity $v_3(\mathbf{x},z,t)$. Using local hydrostatic balance, the pressure is eliminated in terms of the thickness of the vertical layer $h(\mathbf{x},t)$. $h(\mathbf{x},t)$ becomes the third dependent dynamical variable for the reduced system.

The evolution equations for $\mathbf{v}(\mathbf{x},t)$ and $h(\mathbf{x},t)$ serve both as a useful model for the dynamics in a thin layer of fluid and as an important ingredient in more complicated models of the whole atmosphere or ocean [2]. In complex models which attempt to represent the full dynamics of the atmosphere, for example, one must add to the basic shallow-water equations additional dynamics to describe radiative transfer, internal waves, cloud formation, relevant chemistry, etc. Whatever the goal of the dynamics of the shallow-water equations, if one is to solve these equations, some form of truncation of the infinite degrees of freedom must be made to progress numerically. Truncations directly in Eulerian or Lagrangian configuration space or in the dual Fourier space fail to preserve all the conservation laws respected by the underlying particle interchange symmetry of the Lagrangian theory which exhibits itself in the conservation of potential vorticity. These latter remarks, of course, apply only when the shallow water flow is inviscid, as we shall assume throughout this paper. We shall have a few remarks to make at the end of this paper about the use of our results for the case with friction.

In this paper we take up the much studied subject of shallow-water equations with the goal of providing a truncation of the degrees of freedom from infinity to a finite number using a method which preserves the maximum number of conserved quantities consistent with this reduction in the number of degrees of freedom. When this number returns to infinity, that is, when the truncation is removed, the theory preserves all the quantities associated with potential vorticity conservation. Our work takes place in the Lagrangian formulation of the theory. The truncation is made in the Fourier space of variables dual to the Lagrangian labels of fluid particles. The algebra associated with the symmetry of particle interchange is altered as part of the truncation, and the finite number of Casimir invariants of the new algebra, which is su(N) and thus familiar, replace the infinite number of conserved quantities following from potential vorticity conservation. In the limit $N\rightarrow\infty$, the usual conserved quantities are recovered.

The methods we use derive from two sources. One is the work by Fairlie and Zachos [3] on finite algebras in string theory and the other is an application of those methods to the two-dimensional Euler equations indepen-

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dently invented by Rouhi [4] and Zeitlin [5]. The latter application may be quite interesting in other geophysical applications where two-dimensional Eulerian flows are studied, but we have not pursued that line of investigation. We have analyzed the shallow-water equations, as presented here, both for their interest as indicated, and also because they have numerous useful formal similarities with internal wave dynamics and with surface wave physics. Our work here is also in planar geometries. The extension to flows on a sphere, while algebraically complicated, is more or less straightforward in concept as seen in the paper of Hoppe [6].

The shallow-water equations and their numerical solution using various truncations have become a subject of renewed interest of late because of the effort to place these equations and their more complex forms on parallel processing machines [7]. The goal of that effort is to build numerically efficient climate models for investigations of very long times (thousands of simulated years) and/or issues requiring very high spatial resolution. We expect that the truncation presented here, which by its formulation preserves as much of the original symmetry as possible of the structure of the shallow-water equations, will prove an attractive alternative to straightforward finite-element, discrete-spatial-grid, or spectral methods for these equations.

In Sec. II we review the shallow-water equations in Eulerian and Lagrangian formulation and write down the algebraic structure associated with particle interchange symmetry. Section III is devoted to the su(N) truncation of the theory in Lagrangian formulation and also presents the truncated Hamiltonian for the shallow-water flow. Section IV has our comments about further uses of our observations in other problems of geophysical interest and contains the summary of our present work.

II. SHALLOW-WATER THEORY

A. Equations of motion and symmetry

The Eulerian shallow-water equations govern the evolution of a two-dimensional horizontal velocity $\mathbf{v}(\mathbf{x},t)$ in a fluid of vertical thickness $h(\mathbf{x},t)$ via the familiar evolution equations [1]

$$\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t}+\mathbf{v}(x,t)\cdot\nabla\mathbf{v}(x,t)=-g\nabla h(\mathbf{x},t)\ , \tag{2}$$

$$\frac{\partial h(\mathbf{x},t)}{\partial t}+\nabla\cdot[h(\mathbf{x},y)\mathbf{v}(\mathbf{x},t)]=0\ , \tag{3}$$

where g is the gravitational constant and $\nabla=[\partial_1,\partial_2]$ is the horizontal gradient. If the frame is rotating about the z axis at angular velocity $f/2$, a term $\mathbf{v}(\mathbf{x},t)\times\hat{\mathbf{z}}f(\mathbf{x})$ appears in the equation for $\mathbf{v}(x,t)$. As indicated above, these equations follow from the three-dimensional Euler equations of a thin, homogeneous fluid with hydrostatic balance determining the pressure $p(\mathbf{x},z,t)$ in terms of the thickness $p(\mathbf{x},z,t)=g[h(\mathbf{x},t)-z]$.

The total energy

$$H_E(\mathbf{v},h)=\tfrac{1}{2}\int d^2x\,[\,|\mathbf{v}(\mathbf{x},t)|^2+gh(\mathbf{x},t)^2] \tag{4}$$

is conserved by solutions to these equations, and the Eulerian potential vorticity

$$q_E(\mathbf{x},t)=\frac{\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}(x,t)}{h(\mathbf{x},t)} \tag{5}$$

satisfies

$$\frac{\partial q_E(\mathbf{x},t)}{\partial t}+\mathbf{v}(x,t)\cdot q_E(\mathbf{x},t)=0\ . \tag{6}$$

This means that

$$\int d^2x\, h(\mathbf{x},t)G(q_E(\mathbf{x},t)) \tag{7}$$

is time independent for arbitrary $G(q_E)$.

These conservation laws arise from the particle interchange symmetry exhibited by the canonical or Lagrangian formulation of the theory. In Lagrangian formulation, the dynamical variables are the particle position $\mathbf{Y}(\mathbf{r},t)$ at every particle label $\mathbf{r}$ and time and the conjugate momentum $\mathbf{\Pi}(\mathbf{r},t)$. In terms of these variables the Hamiltonian reads

$$H[\mathbf{Y},\mathbf{\Pi}]=\tfrac{1}{2}\int d^2r[\,|\mathbf{\Pi}(\mathbf{r},t)|^2+gJ(\mathbf{Y}(\mathbf{r},t))^{-1}]\ , \tag{8}$$

where the Jacobian

$$J(\mathbf{Y}(\mathbf{r},t))=\frac{\partial(\mathbf{Y})}{\partial(\mathbf{r})}=\frac{\partial Y_1}{\partial r_1}\frac{\partial Y_2}{\partial r_2}-\frac{\partial Y_1}{\partial r_2}\frac{\partial Y_2}{\partial r_1} \tag{9}$$

has been introduced. In this definition it is possible to multiply the Jacobian by an arbitrary positive function of r, which would describe the initial height if the initial conditions are used as labels: $\mathbf{Y}(\mathbf{r},0)=\mathbf{r}$. However, we prefer to absorb this factor into the definition of the labels and use the so-called mass labels [8], in which this factor is set to unity.

The evolution in time of any functional $A[\mathbf{Y}(\mathbf{r},t),\mathbf{\Pi}(\mathbf{r},t)]$ follows from Hamiltonian's equations

$$\frac{\partial A}{\partial t}=\{A,H\}\ , \tag{10}$$

where we have introduced the canonical Poisson brackets between functionals $A[\mathbf{Y},\mathbf{\Pi}]$ and $B[\mathbf{Y},\mathbf{\Pi}]$,

$$\{A[\mathbf{Y},\mathbf{\Pi}],B[\mathbf{Y},\mathbf{\Pi}]\}=\int d^2r\left[\frac{\delta A[\mathbf{Y},\mathbf{\Pi}]}{\delta\mathbf{Y}(\mathbf{r},t)}\cdot\frac{\delta B[\mathbf{Y},\mathbf{\Pi}]}{\delta\mathbf{\Pi}(\mathbf{r},t)}-\frac{\delta B[\mathbf{Y},\mathbf{\Pi}]}{\delta\mathbf{Y}(\mathbf{r},t)}\cdot\frac{\delta A[\mathbf{Y},\mathbf{\Pi}]}{\delta\mathbf{\Pi}(\mathbf{r},t)}\right]\ . \tag{11}$$

The Poisson brackets among the canonical variables $\mathbf{Y}(\mathbf{r},t)$ and $\mathbf{\Pi}(\mathbf{r},t)$ is then given by

$$\{Y_\alpha(\mathbf{r},t),\Pi_\beta(\mathbf{r}',t)\}=\delta_{\alpha\beta}\delta^2(\mathbf{r}-\mathbf{r}') \tag{12}$$

where $\alpha,\beta=1,2$. Using the Hamiltonian (8) and the canonical bracket (11), we find that the equations of motion in the Lagrangian specification are given by

$$\begin{aligned}\frac{\partial Y_\alpha(\mathbf{r},t)}{\partial t}&=\frac{\delta H}{\delta\Pi_\alpha(\mathbf{r},t)}=\Pi_\alpha(\mathbf{r},t)\ ,\\ \frac{\partial\Pi_\alpha(\mathbf{r},t)}{\partial t}&=-\frac{\delta H}{\delta Y_\alpha(\mathbf{r},t)}=-\epsilon_{\alpha\beta}\frac{g}{2}\{J^{-2},Y_\beta(\mathbf{r},t)\}_{\mathbf{r}}\ ,\end{aligned} \tag{13}$$

where we have used the following notation for the Jacobian:

$$\{f,g\}_{\mathbf{r}}=\frac{\partial(f,g)}{\partial(r_1,r_2)}\ , \tag{14}$$

and repeated indices (in this case β) are summed over. $\epsilon_{\alpha\beta}$ is the completely antisymmetric symbol in two dimensions. The reason for the use of this notation will become clear later. The equations of motion in the Lagrangian specification can readily be shown to be equivalent to the more usual Eulerian specification. See Abarbanel and Holm [9]. We just mention here that J^{-1} turns out to be fluid height, and can be shown to satisfy the continuity equation (3), while the actual Lagrangian equations of motion (13) are equivalent to the Eulerian momentum equation (2). Of course the time derivatives in the Lagrangian representation are taken with the label $\mathbf{r}$ held fixed and thus are equivalent to the "total" or advective derivative in the Eulerian representation. Also note that from the first equation it can be seen that the canonical momentum is simply the fluid particle velocity.

The potential vorticity in the Lagrangian coordinates takes the form

$$q(\mathbf{r},t)=\{\Pi_\alpha(\mathbf{r},t),Y_\alpha(\mathbf{r},t)\}_{\mathbf{r}}\ . \tag{15}$$

Since, as noted above, time derivatives in the Lagrangian representation are equivalent to advective derivatives in the Eulerian representation, we now simply have the following conservation law:

$$\frac{\partial q(\mathbf{r},t)}{\partial t}=0\ . \tag{16}$$

This can of course also be proven directly using the equations of motion (13). The integrals

$$C[\mathbf{Y},\mathbf{\Pi}]=\int d^2r\, G(q(\mathbf{r},t)) \tag{17}$$

are clearly constant in time for any $G(q)$. Therefore, the Eulerian conservation laws follow from the Lagrangian conservation law.

B. The potential vorticity algebra

The number of conserved quantities in the Lagrangian specification is clearly infinite since the potential vorticity is conserved for each value of $\mathbf{r}$. The existence of these conserved quantities signals the existence of a symmetry group in this problem, and the conserved quantities are generators of this symmetry in a sense which we will spell out in great detail below. These ideas will be well known to the reader familiar with Hamiltonian theory; however, our applications will be rather novel and it will be useful to discuss them from the point of view of the problem at hand, and also for the benefit of the reader whose memory of Hamiltonian theory is rusty.

In order to study the symmetry group of the shallow-water equations in greater detail, it will be useful to take boundary conditions in $\mathbf{r}$ space to be periodic in a square of size $L\times L$, for all quantities of interest, and work in the Fourier representation. We introduce Fourier series as follows: for functions $f(\mathbf{r})$ we write

$$f(\mathbf{r})=\sum_{\mathbf{n}=-\infty}^{\infty} g(\mathbf{n})\exp[i\kappa\mathbf{n}\cdot\mathbf{r}] \tag{18}$$

and the inverse

$$g(\mathbf{n})=\frac{1}{L^2}\int d^2r\, f(\mathbf{r})\exp[-i\kappa\mathbf{n}\cdot\mathbf{r}]\ , \tag{19}$$

where the vectors $\mathbf{n},\mathbf{m},\ldots$ are composed of integers $\mathbf{n}=[n_1,n_2]$; $n_\alpha=0,\pm1,\pm2,\ldots$ and $\kappa=2\pi/L$. We take the Fourier decomposition of the canonical variables to be

$$\begin{aligned}\mathbf{Y}(\mathbf{r},t)&=\frac{1}{L}\sum_{\mathbf{n}}\mathbf{Q}(\mathbf{n},t)\exp[i\kappa\mathbf{n}\cdot\mathbf{r}]\ ,\\ \mathbf{\Pi}(\mathbf{r},t)&=\frac{1}{L}\sum_{\mathbf{n}}\mathbf{P}(\mathbf{n},t)\exp[i\kappa\mathbf{n}\cdot\mathbf{r}]\ ,\end{aligned} \tag{20}$$

which gives the Poisson brackets in Fourier space

$$\{Q_\alpha(\mathbf{n}),P_\beta(\mathbf{m})\}=\delta_{\alpha\beta}\delta_{\mathbf{n},-\mathbf{m}}\ . \tag{21}$$

Note that in our notation we will often suppress the time dependence of our dynamical variables, especially when we are stressing their role as coordinates on phase space rather than their evolution in time. Using this Fourier transformed set of canonical variables we have for the potential vorticity

$$q(\mathbf{r})=\frac{(2\pi)^2}{L^4}\sum_{\mathbf{n}}\zeta(\mathbf{n})\exp[i\kappa\mathbf{n}\cdot\mathbf{r}]\ , \tag{22}$$

with

$$\begin{aligned}\zeta(\mathbf{n})&=\sum_{\mathbf{m}}\mathbf{n}\times\mathbf{m}P_\alpha(\mathbf{m})Q_\alpha(\mathbf{n}-\mathbf{m})\\ &=\sum_{\mathbf{m},\mathbf{m}'}\mathbf{m}'\times\mathbf{m}P_\alpha(\mathbf{m})Q_\alpha(\mathbf{m}')\delta_{\mathbf{n},\mathbf{m}+\mathbf{m}'}\ .\end{aligned} \tag{23}$$

We have defined the quantity $\mathbf{n}\times\mathbf{m}=n_1m_2-n_2m_1$ in this equation, and the normalization for $\zeta(\mathbf{n})$ has been chosen to make the last formula and many to follow as simple as possible. Now since $q(\mathbf{r})$ is conserved, so are all of its Fourier coefficients,

$$\frac{d\zeta(\mathbf{n})}{dt}=\{\zeta(\mathbf{n}),H\}=0\ . \tag{24}$$

In order to set the stage for what follows we now make some comments concerning the $\zeta(\mathbf{n})$. The ideas introduced will be explained in further detail in the following sections. The $\zeta(\mathbf{n})$ can be taken to form a basis for the generators of the symmetry, in the following sense. Compute the following Poisson brackets, using $\zeta(\mathbf{n})$ as "Hamiltonian" and generating a motion in phase space which we will parametrize by a parameter ϵ

$$\begin{aligned}\frac{\partial\mathbf{Y}(\mathbf{r},\epsilon)}{\partial\epsilon}&=\{\mathbf{Y}(\mathbf{r},\epsilon),\zeta(n)\}\\ &=\frac{1}{2\pi\kappa}\{\mathbf{Y}(\mathbf{r},\epsilon),\exp(-i\kappa\mathbf{n}\cdot\mathbf{r})\}_{\mathbf{r}}\ ,\\ \frac{\partial\mathbf{\Pi}(\mathbf{r},\epsilon)}{\partial\epsilon}&=\{\mathbf{\Pi}(\mathbf{r},\epsilon),\zeta(n)\}\\ &=\frac{1}{2\pi\kappa}\{\mathbf{\Pi}(\mathbf{r},\epsilon),\exp(-i\kappa\mathbf{n}\cdot\mathbf{r})\}_{\mathbf{r}}\ .\end{aligned} \tag{25}$$

Consider the first of these expressions. Note that we have different brackets in the first and second equalities. The first is our usual canonical bracket in phase space, while the second is the **r**-space Jacobian following the notation introduced in Eq. (14). Of course this latter is also a "Poisson bracket," since if one treats r_1 as "coordinate" and r_2 as "momentum," then the Jacobian is precisely the Poisson bracket for two-dimensional space (This *partly* accounts for our use of this notation.) This shows that a solution of the above system can be written

$$\mathbf{Y}(\mathbf{r},\epsilon)=(\mathbf{Y}\circ\mathbf{R})(\mathbf{r},\epsilon)=\mathbf{Y}(\mathbf{R}(\mathbf{r},\epsilon)) , \tag{26}$$

where $\mathbf{R}$ is the solution to the differential equation

$$\frac{dR}{d\epsilon}=\frac{1}{2\pi\kappa}\left[\frac{\partial}{\partial R_2},-\frac{\partial}{\partial R_1}\right]\exp(-i\kappa\mathbf{n}\cdot R) , \tag{27}$$

with $\mathbf{R}(\mathbf{r},0)=\mathbf{r}$. This set of two coupled ordinary differential equations is formally just Hamilton's equations in a two-dimensional phase space with (r_1,r_2) as canonical variables and $(1/2\pi\kappa)\exp(-i\kappa\mathbf{n}\cdot\mathbf{r})$ as the Hamiltonian. It is well known that the solution $\mathbf{R}$ preserves area

$$\frac{\partial(\mathbf{R})}{\partial(\mathbf{r})}=1 . \tag{28}$$

All these considerations apply to the canonical momentum $\mathbf{\Pi}$ as well. Now *any* linear combination of the $\zeta(\mathbf{n})$ is also conserved (in particular we should really consider the real combinations), therefore the Hamiltonian in Eq. (27) can be an arbitrary function $\psi=\psi(\mathbf{r})$, since any function can be expanded in terms of Fourier series on the periodic square. We may therefore summarize by saying that the symmetry corresponding to the conservation of the $\zeta(\mathbf{n})$ is

$$\mathbf{Y}\rightarrow\mathbf{Y}\circ\mathbf{R},\quad \mathbf{\Pi}\rightarrow\mathbf{\Pi}\circ\mathbf{R} , \tag{29}$$

where $\mathbf{R}$ must satisfy Eq. (28), but is otherwise arbitrary. This symmetry is often called the *particle relabeling symmetry,* since its physical meaning is precisely a relabeling of particles that leaves the Jacobian (9) fixed. A few comments are in order here. First, the symmetry operation in (29) is easily shown to be a canonical transformation on the $(\mathbf{Y},\mathbf{\Pi})$ variables, consistent with the well-known fact that solutions of Hamiltonian's equations [in this case Eqs. (25)] generate canonical transformation. Second, it can be shown that the shallow-water Hamiltonian (8) is unchanged under the symmetry operation (29). This of course is the necessary condition for the generators of the symmetry $\zeta(\mathbf{n})$ to be conserved. Third, the symmetry operations form a group, since a composition of the transformations $\mathbf{R}'\circ\mathbf{R}$ satisfies (28) if each of $\mathbf{R}$ and $\mathbf{R}'$ do so. We will refer to this group as the *particle relabeling group.* These ideas and their relation to the Eulerian formulation of fluid dynamics for compressible fluids are discussed in further detail, but in more mathematical language, in the papers of Marsden, Ratiu, and Weinstein [10,11].

We need to examine further the algebraic properties of the $\zeta(\mathbf{n})$. This is best done in the Fourier representation. The following brackets may be computed:

$$\{\zeta(\mathbf{n}),Q_\alpha(\mathbf{m})\}=\mathbf{n}\times\mathbf{m}Q_\alpha(\mathbf{n}+\mathbf{m}) ,$$
$$\{\zeta(\mathbf{n}),P_\alpha(\mathbf{m})\}=\mathbf{n}\times\mathbf{m}P_\alpha(\mathbf{n}+\mathbf{m}) , \tag{30}$$

and

$$\{\zeta(\mathbf{n}),\zeta(\mathbf{m})\}=\mathbf{n}\times\mathbf{m}\zeta(\mathbf{n}+\mathbf{m}) . \tag{31}$$

The last Poisson bracket exhibits the structure of the particle interchange algebra and shows it be an infinite-dimensional, noncommutative algebra, as we might expect since it comes from a continuum set of operations on label space $\mathbf{r}$ given by the symmetry operation (29). This set of continuum operations translates via the Fourier transform into a discrete infinity of operations in $\mathbf{n}$ space. Further, the closure of the bracket (31) signals the fact that the corresponding symmetries form a group as we already observed above.

Finally in this section we note that the Jacobian (9) clearly plays a crucial role, so we will Fourier decompose it also, and record some brackets. The actual motivation for doing so will become clear as we proceed.

$$J(\mathbf{Y}(\mathbf{r},t))=\frac{\partial(\mathbf{Y})}{\partial(\mathbf{r})}=\frac{(2\pi)^2}{L^4}\sum_{\mathbf{n}}\rho(\mathbf{n})\exp[i\kappa\mathbf{n}\cdot\mathbf{r}] , \tag{32}$$

which gives

$$\rho(\mathbf{n})=\tfrac{1}{2}\sum_{\mathbf{m},\mathbf{m}'}\mathbf{m}'\times\mathbf{m}\mathbf{Q}(m)\times\mathbf{Q}(\mathbf{m}')\delta_{\mathbf{n},\mathbf{m}+\mathbf{m}'}=\sum_{\mathbf{m}}\mathbf{n}\times\mathbf{m}Q_1(\mathbf{m})Q_2(\mathbf{n}-\mathbf{m}) . \tag{33}$$

The Poisson bracket of $\zeta(n)$ and $\rho(\mathbf{m})$ is then found to be

$$\{\zeta(\mathbf{n}),\rho(\mathbf{m})\}=\mathbf{n}\times\mathbf{m}\rho(\mathbf{m}+\mathbf{n}) . \tag{34}$$

A couple of striking observations here are first, the similarity of structure in the Poisson brackets given in Eqs. (30), (31), and (34), and second, the similarity in the expressions for the Fourier coefficients of the potential vorticity and the Fourier coefficients of the Jacobian as expressed in terms of the Fourier coefficients of the canonical coordinates and momenta, Eqs. (23) and (33). These observations will be used below.

III. TRUNCATION OF THE MODES

Recall that our aim in this paper is the construction of a finite-dimensional approximation of the shallow-water equations, suitable for numerical study, that possesses a large number of conserved quantities, analogous to the $\zeta(\mathbf{n})$ for the full shallow water equations. We will do this in *two steps.* First we will construction a finite-dimensional group that plays the role of the particle relabeling group for the truncated finite-dimensional system, and second, we will construct a Hamiltonian for the truncated system that is invariant under the action of the finite-dimensional symmetry. The generators of the symmetry will then be conserved by the truncated system and will therefore be analogs of the $\zeta(\mathbf{n})$. Moreover, since the finite-dimensional symmetry group, which will turn out to be SU(N), approximates the particle relabeling group in a sense which we will discuss, the conserved generators for the truncated theory will approach the $\zeta(\mathbf{n})$ as $N\to\infty$.

A. Algebra of the truncated modes

Our truncation of the particle interchange algebra is motivated by the idea of limiting the number of Fourier modes, but a simple cutoff on the components of the vectors $\mathbf{n}$ does not respect the Poisson brackets. Suppose we limit each component of our integer vectors $\mathbf{n}=[n_1,n_2]$ to $-M\le n_\alpha \le M$; then the presence of the sum vectors $\mathbf{n}+\mathbf{m}$ in the Poisson brackets of $\zeta(\mathbf{n})$ means that vectors in the range are mapped out of the range. To address this we note that restricting vectors in label space to a box of size $L\times L$ and making the Fourier transform, we have in effect mapped our space onto a torus by implicitly identifying the sides under the assumption of periodicity. If we were to formalize this periodicity by requiring all sums of integer vectors to lie within the range $[-M,M]$ by a modulo or remainder operation, we would still need to deal with the terms $\mathbf{n}\times\mathbf{m}$ that appear in all the Poisson brackets with $\zeta(\mathbf{n})$. These are the so-called structure constants associated with the group properties of particle relabeling invariance, so it is suggestive that modifying them as well would be required to make a consistent theory of truncated Fourier modes. In particular one must address the Jacobi identity which provides the statement that the operations in question do close to form a group.

The set of operations which provides a consistent truncation of the modes comes from changing the definition of the Fourier components of $q(\mathbf{r},t)$ to

$$\zeta_N(\mathbf{n})=\frac{1}{\kappa_N}\sum_{\mathbf{m}=-M}^{M}\sin[\kappa_N\mathbf{n}\times\mathbf{m}]P_\alpha(\mathbf{m})Q_\alpha(\mathbf{n}-\mathbf{m}) \ , \tag{35}$$

where $N=2M+1$, all components $n_\alpha, m_\alpha, \ldots$ are restricted to $[-M,M]$, and $\kappa_N=2\pi/N$. In the limit $M\to\infty$, or equivalently $N\to\infty$, namely as $\kappa_N\to 0$, this definition of the potential vorticity is the same as in the original Fourier transform. The definition of the Fourier coefficients for the Jacobian is modified to

$$\begin{aligned}\rho_N(\mathbf{n})&=\tfrac{1}{2}\sum_{\mathbf{m},\mathbf{m}'=-M}^{M}\frac{1}{\kappa_N}\sin[\kappa_N\mathbf{m}'\times\mathbf{m}]\mathbf{Q}(\mathbf{m})\\ &\qquad\times\mathbf{Q}(\mathbf{m}')\delta_{\mathbf{n},\mathbf{m}+\mathbf{m}'}\\ &=\sum_{\mathbf{m}=-M}^{M}\frac{1}{\kappa_N}\sin[\kappa_N\mathbf{n}\times\mathbf{m}]Q_1(\mathbf{m})Q_2(\mathbf{n}-\mathbf{m}) \ .\end{aligned} \tag{36}$$

The definitions of the $\mathbf{Q}(\mathbf{n})$ and $\mathbf{P}(\mathbf{n})$ are unchanged and the Poisson brackets between them are still

$$\{Q_\alpha(\mathbf{n}),P_\beta(\mathbf{m})\}=\delta_{\alpha\beta}\delta_{0,\mathbf{m}+\mathbf{n}} \ , \tag{37}$$

with the rule that vector components out of $[-M,M]$ *are mapped back into the range.*

Now the Poisson brackets among the $\zeta_N(\mathbf{n})$ and the other variables $Q_\alpha(\mathbf{n})$, $P_\alpha(\mathbf{n})$, and $\rho_N(\mathbf{n})$ are found to be

$$\begin{aligned}\{\zeta_N(\mathbf{n}),Q_\alpha(\mathbf{m})\}&=-\frac{1}{\kappa_N}\sum_{\mathbf{m}'=-M}^{M}\sin[\kappa_N\mathbf{n}\times\mathbf{m}']\\ &\qquad\times\delta_{0,\mathbf{m}'+\mathbf{m}}Q_\alpha(\mathbf{n}-\mathbf{m}')\\ &=\frac{1}{\kappa_N}\sin[\kappa_N\mathbf{n}\times\mathbf{m}]Q_\alpha(\mathbf{n}+\mathbf{m}) \ ,\end{aligned} \tag{38}$$

and

$$\{\zeta_N(\mathbf{n}),P_\alpha(\mathbf{m})\}=\frac{1}{\kappa_N}\sin[\kappa_N\mathbf{n}\times\mathbf{m}]P_\alpha(\mathbf{n}+\mathbf{m}) \ , \tag{39}$$

$$\{\zeta_N(\mathbf{n}),\rho_N(\mathbf{m})\}=\frac{1}{\kappa_N}\sin[\kappa_N\mathbf{n}\times\mathbf{m}]\rho_N(\mathbf{n}+\mathbf{m}) \ , \tag{40}$$

$$\{\zeta_N(\mathbf{n}),\zeta_N(\mathbf{m})\}=\frac{1}{\kappa_N}\sin[\kappa_N\mathbf{n}\times\mathbf{m}]\zeta_N(\mathbf{n}+\mathbf{m}) \ . \tag{41}$$

In deriving each of these Poisson bracket relations we have used the trigonometric identity

$$\begin{aligned}&\sin(a[\mathbf{m}\times\mathbf{m}'-\mathbf{m}\times\mathbf{n}])\sin(a[\mathbf{n}\times\mathbf{m}'])\\ &\quad+\sin(a[\mathbf{n}\times\mathbf{m}-\mathbf{n}\times\mathbf{m}'])\\ &\qquad=\sin(a[\mathbf{n}\times\mathbf{m}])\sin(a[(\mathbf{n}+\mathbf{m})\times\mathbf{m}]) \ ,\end{aligned} \tag{42}$$

correct for an arbitrary (complex) constant a. For us $a=\kappa_N$.

This set of Poisson brackets defines a finite algebra which is su(N) with the $\zeta_N(\mathbf{n})$ as generators of infinitesimal su(N) transformations. This statement is far from obvious and we will verify it in detail later. For now we note that the truncation of the Fourier modes with the modification of the Poisson brackets provides a consistent reduction from an infinite number of modes to the finite number N. The critical issue in checking this consistency is verifying that the Jacobi identity among Poisson brackets is satisfied, and with the change of structure constants from $\mathbf{n}\times\mathbf{m}\to(1/\kappa_N)\sin[\kappa_N\mathbf{n}\times\mathbf{m}]$ this is readily established.

Next we want to construct a Hamiltonian H_N, which is invariant under this SU(N) and becomes the shallow-water Hamiltonian in the limit $N\to\infty$. The truncated Fourier coefficients are not the most convenient variables in which to construct this Hamiltonian. This is in part due to the form of the potential-energy term in the Hamiltonian (8), which assumes a rather simple form in terms of the fluid particle positions $\mathbf{Y}(\mathbf{r},t)$, but which would look quite complicated if expressed in terms of the Fourier coefficients of $\mathbf{Y}$. This is due to the occurrence of J^{-1}, which involves *inverses* of the $\mathbf{r}$-space gradients of $\mathbf{Y}$. In any case, it is clear that, physically, the natural variables in the Lagrangian representation are the particle positions rather than their Fourier coefficients $Q_\alpha(\mathbf{n})$. We had to introduce the Fourier coefficients to bring out the algebraic structure of the particle relabeling symmetry.

Interestingly enough, it turns out that in the truncated theory, there exist natural variables analogous to $\mathbf{Y}$, which are far more convenient to work with than the Fourier coefficients. In order to motivate the introduction of these variables, it is necessary to examine in greater detail the interplay between the symmetry group and Hamiltonian structure for symmetric Hamiltonian systems.

Perhaps the most important subclass of Hamiltonian systems with symmetry is the one in which the symmetry acts naturally on the *configuration space* of the system. That is the "new" q's are functions only of the "old" q's under the operation of the symmetry. The action of the symmetry on the full phase space is then determined by the requirement that it lead to canonical transformations. Our problem fits into this category, but before we analyze the situation in our complicated case of interest, we consider from this point of view a familiar example which will serve as a guide to the reader. The reader well versed in the subject of Hamiltonian systems with symmetry may skip this section.

B. Illustrative example: The rotation group

Consider a single particle in a central force field. The position of the particle is given by $\mathbf{x}=(x_1,x_2,x_3)$ and have canonical momenta $\mathbf{p}=(p_1,p_2,p_3)$. The Hamiltonian is given by,

$$H(\mathbf{x},\mathbf{p})=\frac{|\mathbf{p}|^2}{2m}+V(|\mathbf{x}|) \ . \tag{43}$$

It is clear that the symmetry group in this problem is the rotation group in three dimensions SO(3) (we consider only rotation without reflections). Concretely, SO(3) consists of 3×3 orthogonal matrices of determinant 1. SO(3) acts on the configuration space of the particle by the usual rotations of three-dimensional space,

$$\mathbf{x}\rightarrow R\mathbf{x} \tag{44}$$

for $\mathbf{x}\in\mathbb{R}^3$ and $R\in\mathrm{SO}(3)$. As we said before, the transformation is now made to act on the full phase space, by demanding that it be canonical (the precise requirement is the differential form $\mathbf{p}\cdot d\mathbf{x}$ be preserved). This gives the full symmetry action as

$$(\mathbf{x},\mathbf{p})\rightarrow(R\mathbf{x},R\mathbf{p}) \ . \tag{45}$$

It is evident that the Hamiltonian (43) is invariant under such a transformation, since R preserves lengths.

For our purposes it is important to establish the relationship between the *infinitesimal generators of the symmetry,* which are the conserved quantities, and the corresponding *infinitesimal generators of the group,* i.e., the *Lie algebra* of the symmetry group. This infinitesimal generators of the symmetry are precisely the three components of the angular momentum,

$$\begin{aligned} L_1&=x_2p_3-x_3p_2 \ , \\ L_2&=x_3p_1-x_1p_3 \ , \\ L_3&=x_1p_2-x_2p_1 \ , \end{aligned} \tag{46}$$

and they are conserved

$$\frac{dL_j}{dt}=\{L_j,H\}=0 \ , \tag{47}$$

where

$$\{A,B\}=\frac{\partial A}{\partial x_j}\frac{\partial B}{\partial p_j}-\frac{\partial B}{\partial x_j}\frac{\partial A}{\partial p_j} \tag{48}$$

is the usual Poisson bracket for functions A and B on $(\mathbf{x},\mathbf{p})$ space. The L_j are generators of rotations in $(\mathbf{x},\mathbf{p})$ space by the same procedure by which the $\zeta(\mathbf{n})$ generated the particle relabeling symmetry on the $(\mathbf{Y}(\mathbf{r}),\mathbf{\Pi}(\mathbf{r}))$ phase space for the shallow water fluid. For example, let θ denote the amount of rotation about the x_3 axis. The appropriate rotation matrix is given by

$$R_3(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{49}$$

Again we generate the symmetry by considering a motion in θ with L_3 as the generator. That is, we consider the differential equations

$$\left[\frac{d\mathbf{x}}{d\theta},\frac{d\mathbf{p}}{d\theta}\right]=(\{\mathbf{x},L_3\},\{\mathbf{p},L_3\}) \ . \tag{50}$$

The solution is given by $\mathbf{x}(\theta)=R_3(\theta)\mathbf{x}(0)$ and $\mathbf{p}(\theta)=R_3(\theta)\mathbf{p}(0)$. In this sense L_3 generates the symmetry in question, i.e., rotation of the $\mathbf{x}$ and $\mathbf{p}$ vectors about the third axis.

However, one may also discuss infinitesimal generators of a group in a way that is entirely independent of any configuration space or phase space the group acts on, after all a group acts also on *itself* by the group multiplication operation. In the theory of Lie groups, the vector space of the infinitesimal generators of the Lie group is called the Lie algebra corresponding to the Lie group. For example, the infinitesimal generator of the rotation given by $R_3(\theta)$ is defined by

$$\omega_3=\left.\frac{dR_3(\theta)}{d\theta}\right|_{\theta=0}=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{51}$$

with ω_1 and ω_2 defined in a similar way. A general rotation is generated by a linear combination

$$\omega_{\mathbf{a}}=a_j\omega_j \ , \tag{52}$$

which generates a general rotation in the sense that the matrix exponential $R_{\mathbf{a}}=\exp(\omega_{\mathbf{a}})$ is the orthogonal matrix that rotates by amount $|\mathbf{a}|$ about the direction defined by the unit vector $\mathbf{a}/|\mathbf{a}|$. The Lie algebra corresponding to a Lie group is usually denoted by lower case letters, e.g., we denote the Lie algebra of SO(3) by so(3). Evidently so(3) consists of arbitrary 3×3 antisymmetric matrices, as seen from Eq. (52).

In general, an N-parameter group has a Lie algebra that is an N-dimensional vector space. For the rotation group SO(3), a suitable basis for its Lie algebra are the collection of matrices $\{\omega_1,\omega_2,\omega_3\}$, linear combinations of which generate general rotations as we have seen. In addition to being a vector space, however, the Lie algebra inherits further structure from the group associated with it. Conversely, this structure will essentially determine the corresponding Lie group. This structure can be made concrete by introducing the *structure constants* c_{ijk} of the

Lie algebra. For a matrix group, if $\{e_1, e_2, \ldots, e_N\}$ are a basis for the Lie algebra, then the structure constants are defined through

$$[e_i, e_j] = c_{ijk} e_k \ , \tag{53}$$

where $[e_i, e_j] = e_i e_j - e_j e_i$ is the usual commutator of matrices. A fact immediately worthy of note here is that the commutator of a pair of Lie algebra elements belongs again to the Lie algebra. For example, the commutator of a pair of antisymmetric matrices is again antisymmetric. Therefore the antisymmetric matrices form a Lie algebra, which is the Lie algebra of the rotation group as we have seen. The symmetric matrices, for example, do not form a Lie algebra.

In terms of the ω_j basis, the structure constants of the rotation group can be readily computed and are given by $c_{ijk} = \epsilon_{ijk}$, where ϵ_{ijk} is the completely antisymmetric symbol in three dimensions. Thus

$$[\omega_i, \omega_j] = \epsilon_{ijk} \omega_k \ . \tag{54}$$

The structure constants, even though they are basis dependent, are sufficient to identify the underlying Lie group [with the proviso that Lie groups that are different only at the global level, such as SO(3) (rotations without reflections) and O(3) (rotations with reflections), will have the same Lie algebra and same structure constants].

Given these ideas, it seems reasonable therefore to define a vector-space isomorphism T relating so(3) and the space of angular momentum functions (angular momenta about all possible axes) given by

$$T(\omega_{\mathbf{a}}) = a_i T(\omega_i) = a_i L_i \equiv L_{\mathbf{a}} \ . \tag{55}$$

This association clearly *is* an isomorphism of vector spaces, but it is more than that. As we saw, the angular momentum functions generate certain canonical transformations (rotations) on $(\mathbf{x}, \mathbf{p})$ phase space. Clearly this is a three-dimensional subgroup of the (infinite dimensional) group of canonical transformations on phase space. The association of vector spaces $\omega_{\mathbf{a}} \leftrightarrow L_{\mathbf{a}}$ respects the group structures in SO(3) and the subgroup of canonical transformations generated by the angular momenta L_j, since

$$T([\omega_i, \omega_j]) = \{L_i, L_j\} \ , \tag{56}$$

because $\{L_i, L_j\} = \epsilon_{ijk} L_k$.

This type of situation is not peculiar to the rotation group, but to all Hamiltonian systems with a continuous symmetry group, where the symmetry group acts naturally on the configuration space. This is true both for the shallow-water equations and their SU(N) symmetric truncation. We will now proceed to examine the situation in this problem.

C. A more detailed look at the particle-relabeling symmetry

In the preceding sections, we discussed the potential vorticity algebra as generated by the $\zeta(\mathbf{n})$ in the continuum and by the $\zeta_N(\mathbf{n})$ in the truncated versions of shallow water theory (of course we have not yet written down the truncated version of the shallow-water equations; this will come later on in the paper). Clearly $\zeta_N(\mathbf{n})$ and $\zeta(\mathbf{n})$ are analogous to the angular momenta L_j in the case of the rotation group discussed above. In order to proceed further, we must now find the corresponding Lie algebra elements, i.e., quantities analogous to the matrices ω_j in the case of SO(3).

In the continuum case we almost gave an answer to this question already, but we will now put these matters into the proper context and discuss this case in detail before we go on to the truncated theory. As we discussed previously, the symmetry of interest in the continuum case is the particle interchange symmetry. These are mappings of the periodic square into itself that preserve area. Such mappings are generated by vector fields of the form given on the right-hand side of Eq. (27), which are the divergence-free vector fields. Divergence-free vector fields generate area-preserving mappings in the sense that for $\mathbf{v}(\mathbf{r})$ divergence free $\partial v_i / \partial r_i = 0$, the solution of the differential equation

$$\frac{d\mathbf{r}}{d\epsilon} = \mathbf{v}(\mathbf{r}) \ , \tag{57}$$

considered for all initial conditions in the region of interest (the periodic square in our case), is a one-parameter family of area-preserving maps of the region into itself, parameterized by ϵ. In two dimensions, we are lucky in that a divergence free vector field can be written in terms of a scalar "stream function"; let us label the divergence-free vector field by the corresponding stream function ψ

$$v_\psi^i = \epsilon_{ij} \frac{\partial \psi}{\partial r_j} \ . \tag{58}$$

In general, in order to examine the group structure for a group of transformations on a region, we look at the *Lie bracket* of the generating vector fields. This is defined by

$$[\mathbf{u}, \mathbf{v}] = \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u} \ . \tag{59}$$

The Lie bracket measures in the infinitesimal sense, the noncommutativity of the flows generated by $\mathbf{u}$ and $\mathbf{v}$. For divergence-free vector fields in two dimensions, one can readily show that

$$[\mathbf{u}_\psi, \mathbf{u}_\phi] = -\mathbf{u}_{\{\psi, \phi\}_{\mathbf{r}}} \ , \tag{60}$$

where we have used the notation introduced in Eq. (14). This shows that the correspondence established between divergence-free vector fields and functions (i.e., stream functions) in two dimensions via Eq. (58) respects the algebraic structure that vector fields are endowed with via the Lie bracket and the one for functions, given by the Jacobian or two-dimensional Poisson bracket. In studying the algebra of the area-preserving mappings of the periodic square into itself, we may therefore consider the infinitesimal generators to be periodic functions ψ, which give rise to the mappings via the flow of the corresponding divergence-free vector field (58). (The reader familiar with Hamiltonian theory will recognize that we are simply speaking of canonical transformations on two-dimensional phase space. What may be a bit confusing here is that we are viewing these transformations as the symmetry group of a much larger Hamiltonian system, namely the shallow-water equations.)

Any such ψ can expanded in a basis

$$\psi(\mathbf{r})=\sum_{n=-\infty}^{\infty} c_{\mathbf{n}}\psi_{\mathbf{n}} , \tag{61}$$

with

$$\psi_{\mathbf{n}}=-\frac{1}{\kappa^2}\exp(i\kappa\mathbf{n}\cdot\mathbf{r}) . \tag{62}$$

This normalization is chosen since it results in

$$\{\psi_{\mathbf{n}},\psi_{\mathbf{m}}\}_{\mathrm{r}}=\mathbf{n}\times\mathbf{m}\psi_{\mathbf{n}\times\mathbf{m}} , \tag{63}$$

which is to be compared to the potential vorticity algebra in Eq. (31). The algebraic structures are exactly the same and we have therefore exhibited the analogues of the ω_i for angular momentum theory. That is the analog of the correspondence

$$\omega_i\leftrightarrow L_i \tag{64}$$

for the particle relabeling group is

$$\psi_{\mathbf{n}}\leftrightarrow\zeta(\mathbf{n}) . \tag{65}$$

The Jacobian in Eq. (63) plays for the $\psi_{\mathbf{n}}$ the role that the commutator bracket played for the ω_i in Eq. (54). This is another reason why we used a bracketlike expression for the Jacobian.

D. The generators $\zeta_N(\mathbf{n})$ and their relationship to the group SU(N)

Next we look at the SU(N) truncated theory. Since our symmetry group is now finite dimensional, it is not surprising that the analog of the ω_i for the rotation group will again be matrices. These matrices will form a basis for the Lie algebra of SU(N), which justifies our giving this name to the symmetry of the truncated equations. The matrices in question were introduced by 't Hooft [12] and have been analyzed recently by various authors [3,6,13] and references therein.

These $N\times N$ matrices, *for N odd,* which we denote by $\hat{T}_{\mathbf{n}}$, are N^2 in number and are indexed by the integer pair $\mathbf{n}=(n_1,n_2)$. The $\hat{T}_{\mathbf{n}}$ are defined in terms of a pair of matrices g and h, and a phase ω, which in turn are defined by

$$\omega=\exp(4\pi i/N) ,$$

$$g=\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & & & \omega^{N-1} \end{bmatrix} , \tag{66}$$

$$h=\begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & 1 & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & 0 & & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} .$$

The $\hat{T}_{\mathbf{n}}$ are then given by

$$\hat{T}_{\mathbf{n}}=\frac{iN}{4\pi}\omega^{n_1n_2/2}g^{n_1}h^{n_2} . \tag{67}$$

Note that this definition holds for arbitrary integers n_1 and n_2 (including negative integers); however, since $g^N=h^N=I$, we have $\hat{T}_{\mathbf{n}+\mathbf{a}N}=\hat{T}_{\mathbf{n}}$ for an arbitrary pair $\mathbf{a}=(a_1,a_2)$ of integers. Therefore on the integer lattice $\mathbf{n}=(n_1,n_2)$, we may take $\mathbf{n}$ from any $N\times N$ cell in order to define the N^2 matrices $\hat{T}_{\mathbf{n}}$. We will take $-M\le n_\alpha\le M$, $\alpha=1,2$, that is, the $N\times N$ cell centered at the origin with $N=2M+1$.

From the definition (67) of the $\hat{T}_{\mathbf{n}}$ matrices, one can show that they satisfy the following commutation relations:

$$[\hat{T}_{\mathbf{n}},\hat{T}_{\mathbf{m}}]=\frac{1}{\kappa_N}\sin[\kappa_N(\mathbf{n}\times\mathbf{m})]\hat{T}_{\mathbf{n}+\mathbf{m}} , \tag{68}$$

with $\kappa_N=2\pi/N$, which is precisely the same algebraic structure as the truncated potential vorticity algebra

$$\{\zeta_N(\mathbf{n}),\zeta_N(\mathbf{m})\}=\frac{1}{\kappa_N}\sin[\kappa_N(\mathbf{n}\times\mathbf{m})]\zeta_N(\mathbf{n}+\mathbf{m}) . \tag{69}$$

Therefore we have now found in the $\hat{T}_{\mathbf{n}}$ the counterparts of the ω_i for SO(3) and $\psi_{\mathbf{n}}$ for the particle relabeling group; and we establish the correspondence

$$\hat{T}_{\mathbf{n}}\leftrightarrow\zeta_N(\mathbf{n}) . \tag{70}$$

We reiterate that the $\hat{T}_{\mathbf{n}}$ are a collection of N^2 *matrices,* for N odd, defined in Eq. (67), while the $\zeta_N(\mathbf{n})$ are a collection of N^2 *functions* on (truncated) *phase space,* given in Eq. (35).

We now come to the question of why the algebra of the $\hat{T}_{\mathbf{n}}$, given in Eq. (68), is that of SU(N). SU(N) consists of the unitary matrices of determinant 1. That is, for $\hat{U}\in\mathrm{SU}(N)$, we have $\hat{U}^\dagger\hat{U}=I$ and $\det\hat{U}=1$, where the dagger denotes the transpose of the complex conjugate. A member $\hat{A}$ of su(N), the Lie algebra of SU(N), must therefore satisfy $\exp(\hat{A})\in\mathrm{SU}(N)$. It can be readily shown that this is true, if and only if $\hat{A}$ is traceless and anti-Hermitian, $\mathrm{tr}(\hat{A})=0$ and $\hat{A}^\dagger=-\hat{A}$. The dimension of su(N) is N^2-1, since a general complex $N\times N$ matrix has $2N^2$ parameters on which the anti-Hermiticity condition places N^2 constraints, and tracelessness provides an additional constraint.

Now the collection $\hat{T}_{\mathbf{n}}$, of N^2 matrices, are linearly independent and traceless for $\mathbf{n}\neq(0,0)$. However, $\hat{T}_{(0,0)}$ plays a trivial role in the algebra (68), since it commutes with all the $\hat{T}_{\mathbf{n}}$ [corresponding to the fact that $\zeta_N(0,0)=0$ identically]. Therefore if the $\hat{T}_{\mathbf{n}}$ for $\mathbf{n}\neq0$ were anti-Hermitian, they would form a basis for the Lie algebra of SU(N). However, the $\hat{T}_{\mathbf{n}}$ are *not* anti-Hermitian. Instead, they satisfy the following relation under the Hermitian conjugate:

$$\hat{T}_{\mathbf{n}}^\dagger=-\hat{T}_{-\mathbf{n}} . \tag{71}$$

As we shall now argue, this *is* precisely the relationship we would want. The reason has to do with the requirement of *reality* on our dynamical variables and other quantities of interest. The Fourier coefficients $Q_\alpha(\mathbf{n})$ and $P_\alpha(\mathbf{n})$ must satisfy $Q^*_\alpha(\mathbf{n})=Q_\alpha(-\mathbf{n})$ and $P^*_\alpha(\mathbf{n})=P_\alpha(-\mathbf{n})$, where the asterisk denotes complex conjugation, in order for the particle positions and momenta $(\mathbf{Y},\mathbf{\Pi})$ to be real. From this, and the definition (35) of the $\zeta_N(\mathbf{n})$, we can readily deduce that $\zeta^*_N(\mathbf{n})=-\zeta_N(-\mathbf{n})$. Since we want real conserved quantities, we form the real combinations of the $\zeta_N(\mathbf{n})$, given by

$$\zeta_N(\mathbf{n})-\zeta_N(-\mathbf{n}),\quad i[\zeta_N(\mathbf{n})+\zeta_N(-\mathbf{n})] \ . \tag{72}$$

Forming analogous combinations of the $\hat{T}_\mathbf{n}$, we have

$$\hat{T}_\mathbf{n}-\hat{T}_{-\mathbf{n}},\quad i(\hat{T}_\mathbf{n}+\hat{T}_{-\mathbf{n}}) \ , \tag{73}$$

which are N^2-1 linearly independent, anti-Hermitian, and [for $\mathbf{n}\neq(0,0)$] traceless matrices. Thus they do form a basis for the Lie algebra of SU(N). We could write the brackets (69) and (68) in terms of these real and antihermitian combinations, respectively, but we will refrain from doing so as it is much more convenient, algebraically, to work with the $\zeta_N(\mathbf{n})$ and $\hat{T}_\mathbf{n}$. This is analogous to the fact it is algebraically simpler to expand functions on the periodic square in term of complex exponentials rather than sines and cosines.

E. Natural variables for the SU(N) symmetric truncation

The preceding discussion would be of little practical value, except that it leads to natural variables for the SU(N) theory. The procedure used is not a standard one in Hamiltonian theory, as far as the authors are aware. The goal is to construct quantities analogous to the particle positions and momenta $(\mathbf{Y},\mathbf{\Pi})$ in the SU(N) truncated theory. When we examine the Fourier decomposition of $\mathbf{Y}$ in Eq. (20), we observe that it may be looked at as a sum of generators of the particle relabeling symmetry, since this is one way to look at the Fourier basis functions, as we saw. Written in this form, the particle positions are given by

$$Y_\alpha(\mathbf{r},t)=-\frac{\kappa^2}{L}\sum_\mathbf{n} Q_\alpha(\mathbf{n},t)\psi_\mathbf{n} \ , \tag{74}$$

with $\psi_\mathbf{n}$ just a scaled complex exponential given in Eq. (62). Now as we observed in Sec. III D, in the truncated theory the matrices $\hat{T}_\mathbf{n}$ play the role of the $\psi_\mathbf{n}$. This motivates the introduction of the following anti-Hermitian *matrices* $\hat{Y}_\alpha$ and $\hat{\Pi}_\alpha$:

$$\hat{Y}_\alpha=\sqrt{\kappa_N^3/\pi}\sum_\mathbf{n} Q_\alpha(\mathbf{n})\hat{T}_\mathbf{n} \ , \tag{75}$$

$$\hat{\Pi}_\alpha=\sqrt{\kappa_N^3/\pi}\sum_\mathbf{n} P_\alpha(\mathbf{n})\hat{T}_\mathbf{n} \ , \tag{76}$$

with the sums now in the range $-M\le n_\alpha\le M$. They are anti-Hermitian because of the behavior of the Fourier coefficients and the $\hat{T}_\mathbf{n}$ under complex conjugation and Hermitian transposition, respectively. Thus $\hat{Y}_\alpha$ and $\hat{\Pi}_\alpha$ are linear combinations of the $\hat{T}_\mathbf{n}$ matrices, with the coefficients being the truncated Fourier coefficients. The normalizations are chosen to simplify the formulas that follow, and are of no special significance.

We may think of Eq. (75) as defining a linear transformation relating the truncated Fourier coefficients $Q_\alpha(\mathbf{n})$ and elements of the matrix $\hat{Y}_\alpha$. These matrix elements we will denote by $\hat{Y}_\alpha(j_1,j_2)$, where $1\le j_\alpha\le N$, for $\alpha=1,2$. This linear transformation is invertible since the $\hat{T}_\mathbf{n}$ are linearly independent. Moreover, $Q_\alpha(\mathbf{n})$ can be recovered explicitly from $\hat{Y}_\alpha$ by the formula,

$$Q_\alpha(\mathbf{n})=-2\sqrt{\kappa_N^3/\pi}\,\mathrm{tr}(\hat{Y}_\alpha\hat{T}_{-\mathbf{n}}) \ , \tag{77}$$

this follows from the following orthogonality relation of the $\hat{T}_\mathbf{n}$ under the trace inner product for matrices,

$$\mathrm{tr}(\hat{T}_\mathbf{n}\hat{T}_\mathbf{m})=-\frac{\pi}{2\kappa_N^3}\delta_{\mathbf{n},-\mathbf{m}} \ . \tag{78}$$

These observations of course apply to $\hat{\Pi}_\alpha$ as well.

We see that instead of the truncated Fourier coefficients $Q_\alpha(\mathbf{n})$, $P_\alpha(\mathbf{n})$, and $-M\le n_\alpha\le M$, we may use the four anti-Hermitian matrices $\hat{Y}_\alpha,\hat{\Pi}_\alpha,\alpha=1,2$ as dynamical variables. We shall soon see that these are far more convenient than the truncated Fourier coefficients. Before we proceed, we note that instead of $4N\times N$ anti-Hermitian matrices, we may use two general complex matrices $\hat{Y}$ and $\hat{\Pi}$ defined by

$$\hat{Y}=\hat{Y}_1+i\hat{Y}_2,\quad \hat{\Pi}=\hat{\Pi}_1+i\hat{\Pi}_2 \ , \tag{79}$$

from which $\hat{Y}_\alpha$ and $\hat{\Pi}_\alpha$ can be recovered through

$$\hat{Y}_1=\tfrac{1}{2}(\hat{Y}-\hat{Y}^\dagger),\quad \hat{Y}_2=\frac{1}{2i}(\hat{Y}+\hat{Y}^\dagger) \ , \tag{80}$$

with a similar formula for $\hat{\Pi}_\alpha$. In what follows, the use of $\hat{Y}$ and $\hat{\Pi}$ will give our formulas a somewhat more compact form.

Recall that our aim is to find an approximate Hamiltonian that is invariant under the action of the SU(N) symmetry. This Hamiltonian will then produce a truncation of the shallow-water equations which preserves the N^2-1 generators $\zeta_N(\mathbf{n})$. The matrix formulation in terms of $\hat{Y}$ and $\hat{\Pi}$ is ideally suited to this task.

We examine now the infinitesimal action of the generators $\zeta_N(\mathbf{n})$ on our new phase-space coordinates $\hat{Y}$ and $\hat{\Pi}$. We therefore consider

$$\{\hat{Y},\zeta_N(\mathbf{n})\}=\sqrt{\kappa_N^3/\pi}\sum_\mathbf{m}\{Q(\mathbf{m}),\zeta_N(\mathbf{n})\}\hat{T}_\mathbf{m} \ , \tag{81}$$

where we have defined $Q(\mathbf{m})=Q_1(\mathbf{m})+iQ_2(\mathbf{m})$. Using the Poisson bracket (38) and the commutation relationship (68) for the $\hat{T}_\mathbf{n}$ matrices, we arrive at the following remarkable formula:

$$\{\hat{Y},\zeta_N(\mathbf{n})\}=[\hat{T}_{-\mathbf{n}},\hat{Y}] \ . \tag{82}$$

Now consider the following quantity:

$$\zeta=\sum_\mathbf{n} c_\mathbf{n}\zeta_N(\mathbf{n}) \ . \tag{83}$$

If $c^*_\mathbf{n}=-c_{-\mathbf{n}}$, then the expression for ζ above represents the most general *real* generator of our SU(N) symmetry. We then have, using Eq. (82),

$$\{\hat{Y},\zeta\}=[\hat{A},\hat{Y}]\ , \tag{84}$$

where

$$\hat{A}=\sum_{\mathbf{n}} c_{\mathbf{n}}\hat{T}_{-\mathbf{n}} \tag{85}$$

is easily seen to be anti-Hermitian.

Thus the infinitesimal action of SU(N) on our configuration space as coordinatized by the matrix $\hat{Y}$ is given by $\hat{Y}\rightarrow[\hat{A},\hat{Y}]$, where $\hat{A}$ is an arbitrary anti-Hermitian matrix. In order to obtain the full symmetry operation corresponding to this infinitesimal action, we must solve the differential equation

$$\frac{d\hat{Y}}{d\theta}=[\hat{A},\hat{Y}]\ . \tag{86}$$

The solution is given by

$$\hat{Y}(\theta)=\hat{U}\hat{Y}(0)\hat{U}^{-1}\ , \tag{87}$$

where $\hat{U}=\exp(\theta\hat{A})$. $\hat{U}$ is unitary since $\hat{A}$ is anti-Hermitian. Additionally, $\hat{A}$ may be taken to be traceless, since the traceless matrix $\hat{A}-(\mathrm{tr}A)I/N$ results in the same differential equation above as $\hat{A}$ itself. This shows that we can take $\hat{U}\in\mathrm{SU}(N)$. Equation (87) then gives a very concrete form to the symmetry operation of SU(N) on our configuration space,

$$\hat{Y}\rightarrow\hat{U}\hat{Y}\hat{U}^{-1}\ . \tag{88}$$

From Eq. (39), we see that the operation of the symmetry on $\hat{\Pi}$ is exactly the same,

$$\hat{\Pi}\rightarrow\hat{U}\hat{Y}\hat{U}^{-1}\ . \tag{89}$$

Moreover, since the infinitesimal action of the generators $\zeta_N(\mathbf{n})$ on the $\rho_N(\mathbf{n})$ [which are modified, truncated Fourier coefficients in the Fourier expansion of the Jacobian (36)] is the same as the action on the $Q_\alpha(\mathbf{n})$ and $P_\alpha(\mathbf{n})$, Eq. (40), it will be useful to define a matrix $\hat{J}_N$ given by

$$\hat{J}_N=\frac{1}{i\pi N}\left[\frac{2\pi}{L}\right]^4\sum_{\mathbf{n}}\rho_N(\mathbf{n})\hat{T}_{\mathbf{n}}\ , \tag{90}$$

where the normalization is chosen for the future convenience. Clearly $\hat{J}_N$ is going to play the role of the Jacobian J for the truncated theory. Note that $\hat{J}_N$ is Hermitian. The action of the SU(N) symmetry on $\hat{J}_N$ is of course again

$$\hat{J}_N\rightarrow\hat{U}\hat{J}_N\hat{U}^{-1}\ . \tag{91}$$

F. The SU(N) symmetric approximate Hamiltonian

We will now use the observations of the previous section in order to construct a Hamiltonian that approximates the full shallow-water Hamiltonian, given in Eq. (8), and is at the same time an invariant of the SU(N) action.

We exhibit first a class of invariant functions of $\hat{Y}$, $\hat{\Pi}$, and $\hat{J}_N$, and then choose a member of this class that approximates the full shallow-water Hamiltonian. Let $F=F(\hat{Y},\hat{Y}^\dagger,\hat{\Pi},\hat{\Pi}^\dagger,\hat{J}_N)$ by any analytic function of its arguments, then under the SU(N) action described above, F becomes

$$\begin{aligned}F(\hat{Y},\hat{Y}^\dagger,\hat{\Pi},\hat{\Pi}^\dagger,\hat{J}_N)\rightarrow F(&\hat{U}\hat{Y}\hat{U}^{-1},\hat{U}\hat{Y}^\dagger\hat{U}^{-1},\\ &\times\hat{U}\hat{\Pi}\hat{U}^{-1},\hat{U}\hat{\Pi}^\dagger\hat{U}^{-1},\hat{U}\hat{J}_N\hat{U}^{-1})\\ =\hat{U}F(&\hat{Y},\hat{Y}^\dagger,\hat{\Pi},\hat{\Pi}^\dagger,\hat{J}_N)\hat{U}^{-1}\ ,\end{aligned} \tag{92}$$

where the last equality is seen to hold by expanding F in a Taylor series in its arguments. This shows that

$$\mathrm{tr}F(\hat{Y},\hat{Y}^\dagger,\hat{\Pi},\hat{\Pi}^\dagger,\hat{J}_N)\ \text{ is an SU}(N)\text{ invariant}\ , \tag{93}$$

which follows from the cyclic property of the trace. Our SU(N) invariant Hamiltonian can be chosen from this class. Two further results that we will need are the following:

$$\mathrm{tr}(\hat{\Pi}\hat{\Pi}^\dagger)\sim\int|\mathbf{\Pi}|^2d^2r,\quad N\rightarrow\infty\ , \tag{94}$$

and

$$\frac{L^2}{N}\mathrm{tr}(\hat{J}_N^m)\sim\int J^m d^2r,\quad N\rightarrow\infty\ . \tag{95}$$

The reader will understand that these are special cases of a class of formulas relating integrals of powers of quantities $\mathbf{Y}$, $\mathbf{\Pi}$, and J, to traces of powers of the matrices $\hat{Y}$, $\hat{\mathbf{\Pi}}$, and $\hat{J}_N$, as $N\rightarrow\infty$. In particular the results above may be proven by expressing the integrals on the right-hand sides in terms of the Fourier coefficients and comparing with the result of expressing the left-hand sides in terms of the truncated Fourier coefficients. The only tricky step in this computation involves the use of an identity for a product of $\hat{T}_{\mathbf{n}}$ matrices. We record this identity here for the reader interested in the details of these calculations

$$\hat{T}_{\mathbf{n}_1}\hat{T}_{\mathbf{n}_2}\cdots\hat{T}_{\mathbf{n}_k}=\left[\frac{i}{2\kappa_N}\right]^{k-1}\prod_{\substack{\alpha,\beta\\ \alpha<\beta}}\exp[i\kappa_N(\mathbf{n}_\beta\times\mathbf{n}_\alpha)]\hat{T}_{\mathbf{n}_1+\cdots+\mathbf{n}_k}\ . \tag{96}$$

Now in Eq. (94), the right-hand side is the kinetic energy for the shallow-water equations, so we have a suitable SU(N) invariant approximation for the kinetic energy, given by $\mathrm{tr}(\hat{\Pi}\hat{\Pi}^\dagger)$. In order to approximate the potential energy in the shallow-water Hamiltonian (8), we observe that from Eq. (95), we may write

$$\frac{L^2}{N}\mathrm{tr}(F(\hat{J}_N))\sim\int F(J)d^2r,\quad N\rightarrow\infty \tag{97}$$

provided F is an analytic function of its argument. We will of course choose $F(x)=x^{-1}$. The reader may object that this function is not analytic, but actually it is analytic everywhere, except at $x=0$, and this precisely is the value that the Jacobian must not assume, on both physical and mathematical grounds. Therefore, in order to apply Eq. (97) we may, for example, think of expanding J about some nonzero average value. This then gives

$$\frac{L^2}{N}\mathrm{tr}(\hat{J}_N^{-1})\sim\int J^{-1}d^2r,\quad N\to\infty\ , \tag{98}$$

which gives us the following SU(N) invariant approximate Hamiltonian:

$$H_N(\hat{Y},\hat{Y}^\dagger,\hat{\Pi},\hat{\Pi}^\dagger)=\mathrm{tr}(\hat{\Pi}\hat{\Pi}^\dagger)+\frac{gL^2}{2N}\mathrm{tr}(\hat{J}_N^{-1})\ . \tag{99}$$

The $\hat{Y}$ dependence of H_N occurs through $\hat{J}_N$, and in fact one can show, using Eqs. (36), (68), and (90), that $\hat{J}_N$ is given by the following commutator:

$$\hat{J}_N=\frac{\pi N^2}{L^4}[\hat{Y},\hat{Y}^\dagger]\ . \tag{100}$$

G. Poisson brackets and the equations of motion

The truncated equations of motion corresponding to our SU(N) invariant Hamiltonian are given by

$$\frac{d\hat{Y}}{dt}=\{\hat{Y},H_N\},\quad \frac{d\hat{\Pi}}{dt}=\{\hat{\Pi},H_N\}\ , \tag{101}$$

with H_N given by Eq. (99). At this point we know the Poisson brackets in terms of the truncated Fourier coefficients $Q_\alpha(\mathbf{n})$ and $P_\alpha(\mathbf{n})$, which are canonical. However, since the transformation between the truncated Fourier coefficients and the elements of the $\hat{Y}$ and $\hat{\Pi}$ matrices is linear, we may readily compute the Poisson bracket relationship among the elements of these matrices, for example,

$$\{\hat{Y}(\mathbf{j}),\hat{\Pi}(\mathbf{k})\}=\sum_{\mathbf{n}}\sum_{\alpha=1,2}\left[\frac{\partial\hat{Y}(\mathbf{j})}{\partial Q_\alpha(\mathbf{n})}\frac{\partial\hat{\Pi}(\mathbf{k})}{\partial P_\alpha(-\mathbf{n})}-\frac{\partial\hat{\Pi}(\mathbf{k})}{\partial Q_\alpha(\mathbf{n})}\frac{\partial\hat{Y}(\mathbf{j})}{\partial P_\alpha(-\mathbf{n})}\right]\ , \tag{102}$$

since $Q_\alpha(\mathbf{n})$ and $P_\alpha(-\mathbf{n})$ are canonically conjugate. $\mathbf{j}$ and $\mathbf{k}$ above index the elements of the $\hat{Y}$ and $\hat{\Pi}$ matrices above, $\mathbf{j}=(j_1,j_2), 1\le j_\alpha\le N$, and similarly for $\mathbf{k}$. The actual computation is tedious and involves detailed use of the properties of the $\hat{T}_\mathbf{n}$ matrices. We just present the results which show, not surprisingly, that the elements of the $\hat{Y}$ and $\hat{\Pi}$ matrices satisfy essentially canonical Poisson bracket relationships

$$\{\hat{Y}(\mathbf{j}),\hat{\Pi}^\dagger(\mathbf{k})\}=\{\hat{Y}^\dagger(\mathbf{j}),\hat{\Pi}(\mathbf{k})\}=\delta_{j_2k_1}\delta_{j_1k_2}\ , \tag{103}$$

$$\{\hat{Y}(\mathbf{j}),\hat{\Pi}(\mathbf{k})\}=0\ . \tag{104}$$

Of course the brackets involving the elements of $\hat{Y}$ only, or the elements of $\hat{\Pi}$ only, also vanish. Using these bracket relations, we can write the truncated equations of motion more concretely as

$$\frac{d\hat{Y}(j_1,j_2)}{dt}=\frac{\partial H_N}{\partial\hat{\Pi}^\dagger(j_2,j_1)}\ , \tag{105}$$

$$\frac{d\hat{\Pi}(j_1,j_2)}{dt}=-\frac{\partial H_N}{\partial\hat{Y}^\dagger(j_2,j_1)}\ . \tag{106}$$

The differentiation in Eq. (106) is a little tricky and must be performed with care. The results can again be neatly expressed in matrix form, giving the following SU(N) symmetric truncated equations of motion:

$$\frac{d\hat{Y}}{dt}=\hat{\Pi},\quad \frac{d\hat{\Pi}}{dt}=\frac{g\pi N}{2L^2}[\hat{J}_N^{-2},\hat{Y}]\ , \tag{107}$$

with $\hat{J}_N$ given in Eq. (100). The evolution of the truncated Fourier coefficients may now be obtained by solving the coupled ordinary differential equations for the $\hat{Y}$ and $\hat{\Pi}$ matrices given above and using the trace formula (77) relating $\hat{Y}$ and $Q_\alpha(\mathbf{n})$ and its analog for $\hat{\Pi}$ and P_α.

H. Conserved quantities of the SU(N) symmetric theory

Our construction of the SU(N) symmetric truncated shallow-water equations (107) guarantees that they preserve the infinitesimal generators $\zeta_N(\mathbf{n})$ of the SU(N) symmetry. It is illuminating, however, to directly verify the conservation laws from the equations of motion. This is done most elegantly by constructing a matrix $\hat{q}_N$ like $\hat{Y}$, $\hat{\Pi}$, and $\hat{J}_N$, but now using the $\zeta_N(\mathbf{n})$ as coefficients for the $\hat{T}_\mathbf{n}$ matrices

$$\hat{q}_N=\frac{\kappa_N^3}{\pi}\sum_{\mathbf{n}}\zeta_N(\mathbf{n})\hat{T}_\mathbf{n}\ . \tag{108}$$

Using the definition (35) of the $\zeta_N(\mathbf{n})$ and the commutation properties of the $\hat{T}_\mathbf{n}$, one can show that $\hat{q}_N$ defined above is expressible in terms of commutators of the $\hat{Y}$ and $\hat{\Pi}$ matrices

$$\hat{q}_N=\tfrac{1}{2}([\hat{Y},\hat{\Pi}^\dagger]+[\hat{Y}^\dagger,\hat{\Pi}])\ . \tag{109}$$

Using this expression it is a simple matter to show directly from the equations of motion (107) that

$$\frac{d\hat{q}_N}{dt}=0\ , \tag{110}$$

which shows that $d\zeta_N(\mathbf{n})/dt=0$ for each $\mathbf{n}$, since the $\hat{T}_\mathbf{n}$ are time independent and linearly independent. Alternatively, it is also easy to see that $\hat{q}_N$ is traceless and anti-Hermitian, therefore (110) does indeed represent N^2-1 conservation laws.

IV. SUMMARY OF THE MAIN RESULTS

We now put down the main results of this paper in one place for the convenience of the reader. In doing this, it is illuminating to write the equations and relations for the shallow-water equations in parallel with the corresponding results for the SU(N) symmetric truncated theory. In order to bring out this parallel further, we define first the complex functions Y and Π on label space $\mathbf{r}$ as

$$\begin{aligned}Y(\mathbf{r},t)&=Y_1(\mathbf{r},t)+iY_2(\mathbf{r},t)\ ,\\ \Pi(\mathbf{r},t)&=\Pi_1(\mathbf{r},t)+i\Pi_2(\mathbf{r},t)\ .\end{aligned} \tag{111}$$

These are clearly analogs of the $\hat{Y}$ and $\hat{\Pi}$ matrices in the truncated theory. In terms of these variables, the continuum shallow-water Hamiltonian (8) is given by

$$H=\frac{1}{2}\int d^2r(\Pi\Pi^*+gJ^{-1}) , \tag{112}$$

where

$$J=\frac{i}{2}\{Y,Y^*\}_r . \tag{113}$$

In the SU(N) symmetric truncated theory, the Hamiltonian is given by

$$H_N=\text{tr}(\hat{\Pi}\hat{\Pi}^\dagger)+\frac{gL^2}{2N}\text{tr}(\hat{J}_N^{-1}) , \tag{114}$$

where

$$\hat{J}_N=\frac{\pi N^2}{L^4}[\hat{Y},\hat{Y}^\dagger] . \tag{115}$$

The fundamental Poisson bracket in the continuum theory is given by

$$\{Y(\mathbf{r}),\Pi^*(\mathbf{r}')\}=\delta^2(\mathbf{r}-\mathbf{r}') , \tag{116}$$

while in the truncated theory we have

$$\{\hat{Y}(\mathbf{j}),\hat{\Pi}^\dagger(\mathbf{k})\}=\delta_{j_2k_1}\delta_{j_1k_2} . \tag{117}$$

This leads to the continuum equations of motion (shallow-water equations)

$$\frac{\partial Y}{\partial t}=\Pi, \quad \frac{\partial\Pi}{\partial t}=\frac{ig}{2}\{J^{-2},Y\}_r \tag{118}$$

while the SU(N) symmetric truncated shallow-water equations are given by

$$\frac{d\hat{Y}}{dt}=\hat{\Pi}, \quad \frac{d\hat{\Pi}}{dt}=\frac{g\pi N}{2L^2}[\hat{J}_N^{-2},\hat{Y}] . \tag{119}$$

In the decomposition into Fourier coefficients, we have for the continuum variables

$$Y(\mathbf{r})=\frac{1}{L}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}Q(\mathbf{n})\exp[i\kappa\mathbf{n}\cdot\mathbf{r}] ,$$
$$\Pi(\mathbf{r})=\frac{1}{L}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}P(\mathbf{n})\exp[i\kappa\mathbf{n}\cdot\mathbf{r}] , \tag{120}$$

where we defined

$$Q(\mathbf{n})=Q_1(\mathbf{n})+iQ_2(\mathbf{n}) ,$$
$$P(\mathbf{n})=P_1(\mathbf{n})+iP_2(\mathbf{n}) . \tag{121}$$

This gives the inverse formula

$$Q(\mathbf{n})=\frac{1}{L}\int d^2r\, Y(\mathbf{r})\exp[-i\kappa\mathbf{n}\cdot\mathbf{r}] , \tag{122}$$

with a similar formula for $P(\mathbf{n})$. For the truncated variables we have the decomposition in terms of truncated Fourier coefficients

$$\hat{Y}=\sqrt{\kappa_N^3/\pi}\sum_{n_1=-M}^{M}\sum_{n_2=-M}^{M}Q(\mathbf{n})\hat{T}_{\mathbf{n}} , \tag{123}$$

$$\hat{\Pi}=\sqrt{\kappa_N^3/\pi}\sum_{n_1=-M}^{M}\sum_{n_2=-M}^{M}P(\mathbf{n})\hat{T}_{\mathbf{n}} , \tag{124}$$

where $N=2M+1$. Here we have the inverse formula

$$Q(\mathbf{n})=-2\sqrt{\kappa_N^3/\pi}\,\text{tr}(\hat{Y}\hat{T}_{-\mathbf{n}}) , \tag{125}$$

with a similar formula for $P(\mathbf{n})$. For the continuum theory we have a real, conserved *field*, the potential vorticity, given by

$$q(\mathbf{r})=\frac{1}{2}(\{\Pi,Y^*\}_r+\{\Pi^*,Y\}_r) . \tag{126}$$

In the truncated theory, we have a conserved traceless, anti-Hermitian *matrix* given by

$$\hat{q}_N=\frac{1}{2}([\hat{Y},\hat{\Pi}^\dagger]+[\hat{Y}^\dagger,\hat{\Pi}]) . \tag{127}$$

The relationship between $q(\mathbf{r})$ and $\hat{q}_N$ may be seen through their decomposition in terms of Fourier coefficients

$$q(\mathbf{r})=\frac{(2\pi)^2}{L^4}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\zeta(\mathbf{n})\exp[i\kappa\mathbf{n}\cdot\mathbf{r}] \tag{128}$$

and

$$\hat{q}_N=\frac{\kappa_N^3}{\pi}\sum_{n_1=-M}^{M}\sum_{n_2=-M}^{M}\zeta_N(\mathbf{n})\hat{T}_{\mathbf{n}} , \tag{129}$$

with $\zeta(\mathbf{n})$ given in Eq. (23) and $\zeta_N(\mathbf{n})$ defined in Eq. (35), the important fact being that $\zeta_N(\mathbf{n})\sim\zeta(\mathbf{n})$ as $N\to\infty$. So that the conservation laws of the truncated theory approach those of the full equations as N becomes large.

V. COMMENTS: USES OF THE SU(N) THEORY

The main achievement of the mode truncation given here of the shallow-water equations in planar geometry is that the truncated theory preserves all parts of potential vorticity conservation consistent with the finite number of modes of the truncated continuum fluid dynamics. This makes the symmetric truncation provided here quite attractive for use in more realistic models based on shallow layers, since the conserved quantities of the full continuum theory are preserved as well as possible by the finite-dimensional approximation to the fluid.

In integrating the inviscid, truncated equations of motion for atmospheric or ocean dynamics, it is important to preserve the full implications of both the SU(N) symmetry and the Hamiltonian structure. The latter is guaranteed by new developments in integrating Hamiltonian systems which go under the name of *symplectic integration* schemes [14–18]. In order to extract the full implications of the symmetry of particle interchange [in its mode truncated SU(N) appearance] one must extend these symplectic integrators to respect the SU(N) symmetry as well. In these papers it is clearly demonstrated that symmetric-integration methods respecting the Hamiltonian dynamics leads to significantly improved numerical results especially for integrations over long times. It is just the latter time scales which are of interest in many contemporary geophysical fluid dynamical problems.

The methods of this paper are extended in two reasonably straightforward ways. One is to spherical geometries [6] with or without rotation. The more interesting extensions are to additional dynamics of geophysical interest. The quasi-two-dimensional dynamics of internal waves and surface gravity waves also possess particle interchange symmetries, and in the finite-dimensional version of these flows, we will also find the methods exhibited here to be of value. We plan to report on both these developments in the near future.

With the variables identified here, one can add friction or viscosity to the evolution equations in the usual more or less *ad hoc* fashion. The conserved quantities of the inviscid theory will, of course, cease to be precisely conserved by the dynamics. Nonetheless, since the inviscid approximation is so good on many large scales of geophysical interest, the symmetries [Hamiltonian and SU(N)] will continue to play an important role in describing observables of significance.

Finally there is an important technical problem still to be solved in the full development of the theory we have presented here. We have given the SU(N) symmetric truncation of the shallow-water equations in *Lagrangian* formulation. While equivalent to a Eulerian version of the same theory, it is not always the best framework in which to perform accurate and efficient numerical work. The development of the Eulerian version of the SU(N) symmetric, finite-mode truncation of the shallow-water theory remains as an interesting challenge. Again we hope to report on this development in the near future.

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